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Sequential BKW-Operators and Function Algebras

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The Korovkin-type approximation theory for function algebras is studied. A complete characterization of the BKW-operators studied by Takahasi is given for function algebras, and this answers Altomare's conjecture affirmatively. As an application, the BKW-operators on the disk algebra for test functions $\{1, z\}$ are determined. © 1996 Academic Press, Inc.

1. INTRODUCTION

In 1953, Korovkin [10] proved an interesting approximation theorem: If $\{T_n\}_{n \in \mathbf{N}}$ is a sequence of positive linear operators on $C([0, 1])$ such that $\|T_n t^j - t^j\|_\infty \to 0$ as $n \to \infty$ for $j = 0, 1$, and 2, then $\|T_n f - f\|_\infty \to 0$ for every $f \in C([0, 1])$ (see also [11]). This theorem says that to prove that a sequence of positive linear operators $\{T_n\}_{n \in \mathbf{N}}$ on $C([0, 1])$ converges strongly to the identity operator, it is sufficient to check the convergence $\|T_n f - f\|_\infty \to 0$ for only three functions $f(t) = 1, t$, and t^2. In 1968, Wulbert [20] strengthened this theorem as follows. Let $C(\Omega)$ be the space of all continuous functions on a compact Hausdorff space Ω. Let S be a subspace of $C(\Omega)$ with $1 \in S$ such that the Choquet boundary of S coincides with Ω. Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a net of bounded linear operators on $C(\Omega)$ such that $\|T_\lambda\| \to 1$ and $\|T_\lambda f - f\|_\infty \to 0$ for each $f \in S$. Then Wulbert theorem says that $\|T_\lambda f - f\|_\infty \to 0$ for every $f \in C(\Omega)$.

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Recently, Takahasi [15–19] has studied bounded linear operators on Banach spaces satisfying a Wulbert-type property, and he called them BKW-operators (see [16]). Let X, Y be normed spaces. We denote by $B(X, Y)$ the set of all bounded linear operators from X into Y. For a given subset S of X, Takahasi denotes by BKW $(X, Y; S)$ the set of all operators T in $B(X, Y)$ satisfying the following BKW condition (BKW is an abbreviation in honor of Bohman [4], Korovkin [10] and Wulbert [20]).

BKW: For every *net* $\{T_\lambda\}_{\lambda \in \Lambda}$ in $B(X, Y)$ such that $\|T_\lambda\| \to \|T\|$ and $\|T_\lambda s - Ts\| \to 0$ for each $s \in S$, it follows that $\|T_\lambda x - Tx\| \to 0$ for every $x \in X$.

Takahasi used nets of operators in the definition of BKW. It is natural to study sequential type BKW-operators as Korovkin's theorem. Hence we denote by s-BKW$(X, Y; S)$ the set of T in $B(X, Y)$ satisfying the following s-BKW condition.

s-BKW: For every *sequence* $\{T_n\}_{n \in \mathbf{N}}$ in $B(X, Y)$ such that $\|T_n\| \to \|T\|$ and $\|T_n s - Ts\| \to 0$ for each $s \in S$, it follows that $\|T_n x - Tx\| \to 0$ for every $x \in X$.

It is not difficult to see that BKW$(X, Y; S) \subset s$-BKW$(X, Y; S)$. When $X = Y$, we write $B(X) = B(X, X)$ and BKW$(X; S) = BKW(X, X; S)$, etc. Then by the Korovkin and Wulbert theorems, $I \in$ BKW$(C([0,1]); \{1, t, t^2\})$ and $I \in$ BKW$(C(\Omega); S)$ for those subsets S whose Choquet boundary coincides with Ω, where I is the identity operator.

Our subject of this paper is to determine all operators in BKW$(X, Y; S)$ and s-BKW$(X, Y; S)$ for a given subset S of X. Another interesting problem is to determine all subsets S of X which satisfy $T \in$ BKW$(X, Y; S)$ for a given operator $T \in$ B(X, Y). In this case, S is called the *Korovkin set for T* (see a recent monograph by Altomare and Campiti [3]). These two problems are essentially the same.

In Section 2, we prove that if S is a separable subset of X then s-BKW$(X, Y; S)$ = BKW$(X, Y; S)$, but generally s-BKW$(X; S) \neq$ BKW$(X; S)$. In Section 3, we give a characterization of all operators T in BKW$(X, A; S)$ for a function algebra A which is a generalization of [19, Theorem 1.4]. This characterization of BKW$(X, A; S)$ gives us an affirmative answer to the conjecture posed by Altomare in [2]. In Section 4, we determine BKW$(\mathscr{A}; \{1, z\})$ for the disk algebra $\mathscr{A}$ and discuss BKW$(\mathscr{A}; \{1, z, z^2\})$.

2. Sequential BKW-Operators

In [17, 19], Takahasi proved the following lemma.

Lemma 1. *Let $T \in B(X, Y)$. Then $T \in$ BKW$(X, Y; S)$ if and only if for every net $\{T_\lambda\}_{\lambda \in \Lambda}$ in $B(X, Y)$ such that $\|T_\lambda\| \leqslant \|T\|$ for each $\lambda \in \Lambda$ and*

$\|T_\lambda s - Ts\| \to 0$ *for each* $s \in S$, *it follows that* $\|T_\lambda x - Tx\| \to 0$ *for every* $x \in X$.

By the same method, we can prove the following. Here we give an elementary proof. We denote by **N** the set of positive integers.

LEMMA 2. *Let* $T \in B(X, Y)$. *Then* $T \in s$-BKW$(X, Y; S)$ *if and only if for every sequence* $\{T_n\}_{n \in \mathbf{N}}$ *in* $B(X, Y)$ *such that* $\|T_n\| \leqslant \|T\|$ *for each* $n \in \mathbf{N}$ *and* $\|T_n s - Ts\| \to 0$ *for each* $s \in S$, *it follows that* $\|T_n x - Tx\| \to 0$ *for every* $x \in X$.

Proof. The proof of sufficiency is trivial. Hence we suppose that $T \in s$-BKW$(X, Y; S)$. When $T = 0$, there is nothing to prove, so we assume that $T \neq 0$. Let $\{T_n\}_{n \in \mathbf{N}}$ be a sequence in $B(X, Y)$ such that

$$\|T_n\| \leqslant \|T\| \qquad \text{for} \quad n \in \mathbf{N}, \tag{1}$$

$$\|T_n s - Ts\| \to 0 \qquad \text{for} \quad s \in S. \tag{2}$$

To prove $\|T_n x - Tx\| \to 0$ for $x \in X$, suppose not. Then there exists $x_0 \in X$ such that

$$\limsup_{n \to \infty} \|T_n x_0 - Tx_0\| \neq 0.$$

By considering a subsequence, we may assume that

$$\lim_{n \to \infty} \|T_n x_0 - Tx_0\| \neq 0. \tag{3}$$

Let

$$G_n = T_n - a_n(T - T_n) = (1 + a_n)\, T_n - a_n\, T, \tag{4}$$

where a_n is a nonnegative number such that $\|G_n\| = \|T\|$. By (1), (3), and the intermediate valued theorem, there exist such a_n and $\{a_n\}_n$ is a bounded sequence. By (4),

$$G_n - T = (1 + a_n)(T_n - T). \tag{5}$$

Then by (2), for each $s \in S$ we have

$$\|G_n s - Ts\| = (1 + a_n)\|T_n s - Ts\| \to 0 \qquad \text{as} \quad n \to \infty.$$

Since $T \in s$-BKW$(X, Y; S)$, $\|G_n x - Tx\| \to 0$ for every $x \in X$. But by (3) and (5), $\|G_n x_0 - Tx_0\| = (1 + a_n)\|T_n x_0 - Tx_0\|$ does not converge to 0. This is a contradiction.

THEOREM 1. *Let S be a separable subset of X. Then s-BKW$(X, Y; S) =$ BKW$(X, Y; S)$.*

Proof. We need to prove that s-BKW$(X, Y; S) \subset$ BKW$(X, Y; S)$. Let $T \in s$-BKW$(X, Y; S)$. Suppose that $T \notin$ BKW$(X, Y; S)$. Then by Lemma 1, there exists a net $\{T_\lambda\}_{\lambda \in \Lambda}$ in $B(X, Y)$ such that

$$\|T_\lambda\| \leqslant \|T\| \qquad \text{for every} \quad \lambda \in \Lambda, \tag{1}$$

$$\|T_\lambda s - Ts\| \to 0 \qquad \text{for} \quad s \in S, \tag{2}$$

and $\|T_\lambda x_0 - Tx_0\|$ does not converge to 0 for some $x_0 \in X$. Then there exists a positive number $\sigma > 0$ such that for each $\lambda_0 \in \Lambda$, we have

$$\|T_\lambda x_0 - Tx_0\| > \sigma \qquad \text{for some} \quad \lambda > \lambda_0. \tag{3}$$

Let $\tilde{S}$ be the closed linear span of S in X. Then $\tilde{S}$ is a separable Banach space. Here we use the fact that the strong operator topology on bounded sets of $B(X, Y)$ is metrizable if X is separable. By (1) and (2), we see that $T_\lambda \to T$ strongly on $\tilde{S}$. Hence by the above fact and (3), there exists a subsequence $\{T_{\lambda_n}\}_{n \in \mathbf{N}}$ in $\{T_\lambda\}_{\lambda \in \Lambda}$ such that $T_{\lambda_n} \to T$ strongly on $\tilde{S}$ and

$$\|T_{\lambda_n} x_0 - Tx_0\| > \sigma \qquad \text{for} \quad n \in \mathbf{N}. \tag{4}$$

Since $T \in s$-BKW$(X, Y; S)$, by Lemma 2 $T_{\lambda_n} \to T$ strongly on X. But this contradicts (4). Thus we get our assertion.

The following example shows that Theorem 1 is not true without the condition of separability of S.

EXAMPLE 1. Let $\Omega = \beta\mathbf{N} \setminus \mathbf{N}$, where $\beta\mathbf{N}$ is the Stone–Čech compactification of $\mathbf{N}$, and let S be the ideal of all continuous functions vanishing at a single point. We note that S is not a separable subspace of $C(\Omega)$. In [13], Scheffold proved that

$$I \notin \text{BKW}(C(\Omega); S) \qquad \text{and} \qquad I \in s\text{-BKW}(C(\Omega); S),$$

where I is the identity operator on $C(\Omega)$; see also [9, p. 164].

3. FUNCTION ALGEBRAS

Let Ω be a compact Hausdorff space. A supremum norm closed subalgebra A of $C(\Omega)$ is called a function algebra on Ω if A contains constant functions, and for distinct points x, y in Ω there exists a function $f \in A$ such that $f(x) \neq f(y)$. In this section, every A denotes a function algebra on Ω. We denote by ∂A the Shilov boundary of A, that is, ∂A is the smallest closed subset of Ω such that $\|f\|_\infty = \sup\{|f(x)|; x \in \partial A\}$ for every $f \in A$. When $A = C(\Omega)$, we have $\partial A = \Omega$. We identify a point ζ in Ω with δ_ζ the point evaluation of A at ζ: $\delta_\zeta(f) = f(\zeta)$ for $f \in A$. Hence we may consider that $\partial A \subset \Omega \subset A^*$. A closed subset E of Ω is called a peak set for A if there exists f in A such that $f = 1$ on E and $|f| < 1$ on $\Omega \backslash E$. A point ζ in Ω is called a p-point if $\{\zeta\} = \bigcap_\alpha E_\alpha$ for some peak sets E_α. The set of p-points is dense in ∂A. Let D be the open unit disk and let Γ be the unit circle. Let $\mathscr{A}$ be the disk algebra, that is, $\mathscr{A}$ is the supremum norm closed algebra of continuous functions on $\bar{D}$ and analytic in D. Then we have $\partial \mathscr{A} = \Gamma$. If $f \in A$ with $\|f\|_\infty \leqslant 1$, then $h \circ f \in A$ for every $h \in \mathscr{A}$. References [5, 7] are nice for function algebras and the disk algebra.

In this section, we study s-BKW$(X, A; S)$ and BKW$(X, A; S)$ for normed spaces X and $S \subset X$. We denote by X^* the dual space of X and $X_r^* = \{F \in X^*; \|F\| \leqslant r\}$ for $r > 0$. Let

$$U_S(X_r^*) = \{F \in X_r^*; \text{ if } G \in X_r^*, F = G \text{ on } S, \text{ then } F = G \text{ on } X\}.$$

The set $U_S(X_r^*)$ is called the uniqueness set for S and this set plays an essential role in the study of the Korovkin type of approximation theorems (see [12, 16, 19]).

LEMMA 3 [19, Theorem 1.2]. *Let X, Y be normed spaces and $S \subset X$. Let E be a weak*-closed subset of Y_1^* such that $\|y\| = \sup\{|F(y)|; F \in E\}$ for every $y \in Y$. Let $T \in B(X, Y)$. If $T^*(E) \subset U_S(X_{\|T\|}^*)$, then $T \in$ BKW$(X, Y; S)$, where T^* is the dual operator of T.*

The main interest is whether the converse assertion of the above result is true or not for a function algebra $Y = A$ and $E = \partial A$. In [19, Theorem 1.4], Takahasi proved that the converse is true when Y is a supremum norm closed subalgebra of continuous functions on a locally compact Hausdorff space which contains the space of all continuous functions having compact support. Therefore

$$\mathrm{BKW}(X, C(\Omega); S) = \{T \in B(X, C(\Omega)); T^*(\Omega) \subset U_S(X_{\|T\|}^*)\}$$

for every normed space X and $S \subset X$, where we identify a point ζ in Ω with the unit point mass δ_ζ at ζ. When $Y = A$ is a function algebra, we have the following theorem.

THEOREM 2. *Let A be a function algebra. Let X be a normed space and $S \subset X$. Then*

$$\mathrm{BKW}(X, A; S) = \{T \in B(X, A); T^*(\partial A) \subset U_S(X_{\|T\|}^*)\}.$$

Moreover, if ∂A *satisfies the first countability axiom, then* s-BKW$(X, A; S)$ $= BKW(X, A; S)$.

COROLLARY 1. *If* Ω *is a compact Hausdorff space satisfying the first countability axiom, then* s-BKW$(X, C(\Omega); S) =$ BKW$(X, C(\Omega); S)$ *for every normed space* X *and* $S \subset X$.

By Example 1, we can not remove the condition of the first countability of Ω in Corollary 1.

Proof of Theorem 2. By Lemma 3, we have

$$\{T \in B(X, A); T^*(\partial A) \subset U_S(X^*_{\|T\|})\} \subset \mathrm{BKW}(X, A; S).$$

To show the converse inclusion, let $T \in$ BKW$(X, A; S)$ and $\zeta_0 \in \partial A$. We may assume $\|T\| = 1$. We shall prove that

$$T^* \zeta_0 = T^* \delta_{\zeta_0} \in U_S(X^*_1).$$

Let $F \in X^*$ such that $\|F\| \leqslant 1$ and $F = T^* \zeta_0$ on S. Then

$$F(s) = (Ts)(\zeta_0) \qquad \text{for} \quad s \in S. \tag{1}$$

It is sufficient to prove that

$$F(x) = (Tx)(\zeta_0) \qquad \text{for every} \quad x \in X. \tag{2}$$

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be the net of all open neighborhoods of ζ_0 in ∂A. Then there exists a p-point ζ_λ in U_λ, and there exists $h_\lambda \in A$ such that $\|h_\lambda\|_\infty = 1$ and

$$\zeta_\lambda \in \{\xi \in \partial A; |h_\lambda(\xi)| = 1\} = \{\xi \in \partial A; h_\lambda(\xi) = 1\} \subset U_\lambda. \tag{3}$$

For each fixed $\lambda \in \Lambda$, using h_λ we shall find a sequence $\{f_{\lambda, n}\}_{n \in \mathbf{N}}$ in A such that

$$f_{\lambda, n}(\zeta_\lambda) = 1 \qquad \text{for every} \quad n \in \mathbf{N}, \tag{4}$$

$$|f_{\lambda, n}| + |1 - f_{\lambda, n}| < 1 + 1/n \qquad \text{on } \partial A, \tag{5}$$

$$|f_{\lambda, n}| < 1/n \qquad \text{on } \partial A \backslash U_\lambda. \tag{6}$$

Let

$$B_n = \{z \in C; |z| + |1 - z| \leqslant 1 + 1/n\} \quad \text{and} \quad \tilde{B}_n = \{z \in C; |z| < 1/n\} \cap B_n. \tag{7}$$

Let

$$g(z) = (1 + z)/2, \qquad z \in \bar{D},$$

and for $0 < r < 1$ let

$$\phi_r(z) = \frac{z - r}{1 - rz}, \qquad z \in \bar{D},$$

$$\psi_r(z) = \frac{\sigma(z)^r - 1}{\sigma(z)^r + 1}, \qquad \text{where} \quad \sigma(z) = \frac{1 + z}{1 - z}, \qquad z \in \bar{D}.$$

Then g, ϕ_r, and ψ_r are contained in the disk algebra $\mathscr{A}$. Since $\psi_r(\bar{D})$ is the lens-shaped closed domain (see [14, p. 27]), by taking r_1 very close to 0 we have

$$g(\psi_{r_1}(\bar{D})) \subset B_n. \tag{8}$$

We note that r_1 depends on n. By (3), $1 \in h_\lambda(U_\lambda)$ and $h_\lambda(U_\lambda^c)$ is a compact subset of D. Hence by taking r_2 very close to 1, we have

$$\phi_{r_2}(h_\lambda(U_\lambda^c)) \subset \psi_{r_1}^{-1}(g^{-1}(\tilde{B}_n)). \tag{9}$$

We note that r_2 depends on n. Let

$$f_{\lambda, n} = g \circ \psi_{r_1} \circ \phi_{r_2} \circ h_\lambda.$$

Since $g \circ \psi_{r_1} \circ \phi_{r_2} \in \mathscr{A}$ and $h_\lambda \in A$, we have $f_{\lambda, n} \in A$. Since $1 = g(1) = \psi_{r_1}(1) = \phi_{r_2}(1)$, by (3) we get (4). By (7) and (8), we have (5). By (7) and (9), we get (6).

Now for $\lambda \in \Lambda$ and $n \in \mathbf{N}$, let

$$T'_{\lambda, n} x = F(x) f_{\lambda, n} + (1 - f_{\lambda, n})\, Tx, \qquad x \in X. \tag{10}$$

Then $T'_{\lambda, n} \in B(X, A)$, and on ∂A we have

$$\begin{aligned} |T'_{\lambda, n} x| &\leqslant |F(x)| |f_{\lambda, n}| + |1 - f_{\lambda, n}| \|Tx\|_\infty \\ &\leqslant (|f_{\lambda, n}| + |1 - f_{\lambda, n}|)\, \|x\| \\ &\leqslant (1 + 1/n) \|x\| \qquad \text{by (5)}. \end{aligned}$$

Hence $\|T'_{\lambda, n}\| \leqslant 1 + 1/n$. Let

$$T_{\lambda, n} = \frac{n}{n+1}\, T'_{\lambda, n}. \tag{11}$$

Then $\|T_{\lambda, n}\| \leqslant 1 = \|T\|$. We note that $\{T_{\lambda, n}\}_{(\lambda, n) \in \Lambda \times \mathbf{N}}$ is a net. We claim that

$$\lim_{\lambda, n} \|T_{\lambda, n} s - Ts\|_\infty = 0 \qquad \text{for} \quad s \in S. \tag{12}$$

To show this, let $s \in S$ with $\|s\| = 1$. By (10) and (11),

$$T_{\lambda, n}x - Tx = \frac{n}{n+1}(F(x) - Tx) f_{\lambda, n} - \frac{1}{n+1} Tx, \qquad x \in X. \tag{13}$$

For any $\varepsilon > 0$, by (1) there exists $\lambda_\varepsilon \in \Lambda$ such that

$$|F(s) - (Ts)(\zeta)| < \varepsilon \qquad \text{for} \quad \zeta \in U_\lambda, \lambda > \lambda_\varepsilon.$$

Then by (5) and (13),

$$|(T_{\lambda, n}s - Ts)(\zeta)| \leqslant \varepsilon + \frac{1}{n+1} \qquad \text{for} \quad \zeta \in U_\lambda, \lambda > \lambda_\varepsilon. \tag{14}$$

By (6) and (13),

$$|(T_{\lambda, n}s - Ts)(\zeta)| < 3/(n+1) \qquad \text{for} \quad \zeta \in \partial A \setminus U_\lambda, \lambda > \lambda_\varepsilon. \tag{15}$$

By (14) and (15), we obtain (12).

Since $T \in \mathrm{BKW}(X, A; S)$, by Lemma 1 we have

$$\lim_{\lambda, n} \|T_{\lambda, n}x - Tx\|_\infty = 0 \qquad \text{for every} \quad x \in X.$$

Hence

$$\lim_{\lambda, n} |(T_{\lambda, n}x - Tx)(\zeta_\lambda)| = 0. \tag{16}$$

By (4) and (13),

$$(T_{\lambda, n}x - Tx)(\zeta_\lambda) = \frac{n}{n+1} F(x) - (Tx)(\zeta_\lambda).$$

Since $\zeta_\lambda \to \zeta_0$, by (16) we get (2). As a consequence, we obtain

$$\mathrm{BKW}(X, A; S) = \{T \in B(X, A);\ T^*(\partial A) \subset U_S(X^*_{\|T\|})\}. \tag{17}$$

When ∂A satisfies the first countability axiom, we can take a sequence of open subsets $\{U_n\}_{n \in \mathbf{N}}$ of ∂A such that

$$\{\zeta_0\} = \bigcap_{n=1}^{\infty} U_n \subset U_{n+1} \subset U_n.$$

Replacing λ by n in the proof of the first assertion, by the same argument we can prove that

$$s\text{-}\mathrm{BKW}(X, A; S) \subset \{T \in B(X, A);\ T^*(\partial A) \subset U_S(X^*_{\|T\|})\}.$$

Then by (17) and $\mathrm{BKW}(X, A; S) \subset s\text{-}\mathrm{BKW}(X, A; S)$, we obtain that $s\text{-}\mathrm{BKW}(X, A; S) = \mathrm{BKW}(X, A; S)$.

When $X = A$, as a special case of Theorem 2 we have the following corollary.

COROLLARY 2. *Let A be a function algebra and $S \subset A$. Then the identity operator is contained in* BKW$(A; S)$ *if and only if for every $\zeta \in \partial A$ and $\mu \in A^*$ such that $\|\mu\| \leqslant 1$ and $\mu(h) = h(\zeta)$ for every $h \in S$, it follows that $\mu = \zeta$.*

This result solves in the positive a conjecture posed by Altomare in [2]. Also, by Theorem 2, we have the following corollary. We note that this corollary also follows from combining Corollary 2 with Altomare's result [1, Theorem 3.1].

COROLLARY 3. *Let A be a function algebra and $S \subset A$. Then the identity operator is contained in* BKW$(A; S)$ *if and only if for every function algebra B, every $T \in B(A, B)$ satisfying $T^*(\partial B) \subset \partial A$ is contained in* BKW$(A, B; S)$.

4. BKW-OPERATORS ON THE DISK ALGEBRA

In this section, we mainly discuss BKW$(\mathscr{A}; S)$ for the disk algebra $\mathscr{A}$. By Theorem 2, for a function algebra A we have $T \in$ BKW$(A; S)$ if and only if $T^*(\partial A) \subset U_S(A^*_{\|T\|})$. Generally, it is difficult to check whether T satisfies the latter condition or not. For $S \subset A$, let $\tilde{S}$ be the closed linear span of S. We have the following lemma (see also [19, Lemma 2.1]).

LEMMA 4. *Let A be a function algebra, $S \subset A$ with $\tilde{S} \neq A$, and $R > 0$. Then $U_S(A^*_R) = U_{\tilde{S}}(A^*_R)$. If $F \in U_S(A^*_R)$, then $\|F\|_{\tilde{S}} = R$, where $\|F\|_{\tilde{S}} =$* $\sup\{|F(s)|; s \in \tilde{S}, \|s\|_\infty = 1\}$.

Proof. It is easy to see that $U_S(A^*_R) = U_{\tilde{S}}(A^*_R)$. Let $F \in U_S(A^*_R)$. To prove $\|F\|_{\tilde{S}} = R$, suppose that $\|F\|_{\tilde{S}} < R$. By the Hahn–Banach extension theorem, there exists $G \in A^*$ such that $G = F$ on $\tilde{S}$ and $\|G\| = \|F\|_{\tilde{S}} < R$. Then there exists $H \in A^*$ such that $H = 0$ on $\tilde{S}$, $H \neq 0$ on A, and $\|H\| < R - \|G\|$. Let

$$G_r = G + rH, \qquad 0 \leqslant r \leqslant 1.$$

Then $\|G_r\| \leqslant R$, $G_r = G = F$ on S, and $G_{r_1} \neq G_{r_2}$ for $r_1 \neq r_2$. Hence $F \notin U_S(A^*_R)$. This is a contradiction.

We may identify A and $A_{|\partial A}$ Then A is a closed subalgebra of $C(\partial A)$. We denote by $M(\partial A)$ the space of bounded (with respect to the total variation norm) Borel measures on ∂A and $M_r(\partial A) = \{\mu \in M(\partial A); \|\mu\| \leqslant r\}$ for $r > 0$. By the Hahn–Banach and Riesz representation theorems, for each $F \in A^*$

there are some measures μ in $M(\partial A)$ such that $\|F\| = \|\mu\|$ and $F(f) = \int_{\partial A} f\, d\mu$ for $f \in A$. By identifying $F \in A^*$ with one of the above measures μ, we consider that

$$A^* \subset M(\partial A) = C(\partial A)^*.$$

Under these considerations, Theorem 2 and Lemma 4 state the following.

LEMMA 5. *Let A be a function algebra and $S \subset A$ with $\tilde{S} \neq A$. Then $T \in \mathrm{BKW}(A; S)$ if and only if for each $\zeta \in \partial A$, it follows that*

(1) $T^*\zeta \in M(\partial A)$ *and* $\|T^*\zeta\| = \|T\|$,

(2) $\|T^*\zeta\| = \sup\{|\int_{\partial A} s\, dT^*\zeta|;\ s \in \tilde{S},\ \|s\|_\infty = 1\}$,

(3) *if* $\mu \in M_{\|T\|}(\partial A)$ *and* $\int_{\partial A} s\, dT^*\zeta = \int_{\partial A} s\, d\mu$ *for* $s \in S$, *then* $\int_{\partial A} f\, dT^*\zeta = \int_{\partial A} f\, d\mu$ *for every* $f \in A$.

In the above, condition (3) is essential, and (1) and (2) are deduced from (3). To clear the condition on $T^*\zeta$ for $T \in \mathrm{BKW}(A; S)$, we add (1) and (2).

Let $\mathscr{A}$ be the disk algebra on $\bar{D}$. Then $\partial\mathscr{A} = \Gamma$, the unit circle. For $\psi \in \mathscr{A}$, let M_ψ be the multiplication operator on $\mathscr{A}$: $M_\psi f = \psi f$ for $f \in \mathscr{A}$. For $\phi \in \mathscr{A}$ with $\|\phi\|_\infty \leqslant 1$, let C_ϕ be the composition operator on $\mathscr{A}$: $C_\phi f = f \circ \phi$ for $f \in \mathscr{A}$. If $b \in \mathscr{A}$ and $|b| = 1$ on Γ, then b is a finite Blaschke product and b has the form

$$b(z) = \lambda \prod_{n=1}^{k} \frac{-\overline{z_n}}{|z_n|} \frac{z - z_n}{1 - \overline{z_n} z}, \qquad z \in \bar{D},$$

for some constant λ with $|\lambda| = 1$ and $z_n \in D$, $n = 1, 2, \ldots, k$, where we consider that $-0/0 = 1$ (see [7]). Since $\Gamma = \partial\mathscr{A}$, by Theorem 1 or 2 we have s-$\mathrm{BKW}(\mathscr{A}; \{1, z\}) = \mathrm{BKW}(\mathscr{A}; \{1, z\})$. In [16], Takahasi proves that $C_\phi \in \mathrm{BKW}(\mathscr{A}; \{1, z\})$ for a finite Blaschke product ϕ. The following theorem gives a complete characterization of operators in $\mathrm{BKW}(\mathscr{A}; \{1, z\})$.

THEOREM 3. *Let $\mathscr{A}$ be the disk algebra. Then*

$$\mathrm{BKW}(\mathscr{A}; \{1, z\}) = \{aM_\psi C_\phi;\ \psi, \phi \text{ are finite Blaschke products}, a \in C\}.$$

Proof. Let $S = \{1, z\}$. Let ψ, $\phi \in \mathscr{A}$ with $|\psi| = |\phi| = 1$ on Γ and let $T = M_\psi C_\phi$. Then $\|T\| = 1$. To show $T \in \mathrm{BKW}(\mathscr{A}; \{1, z\})$, let $\zeta \in \Gamma$. By the definition of T,

$$(T^*\zeta)(f) = (Tf)(\zeta) = \psi(\zeta)(f \circ \phi)(\zeta) = (\psi(\zeta)\, \delta_{\phi(\zeta)})(f)$$

for $f \in \mathscr{A}$. Hence we may consider that $T^*\zeta = \phi(\zeta)\, \delta_{\phi(\zeta)}$. Now it is easy to see that $T^*\zeta \in U_S(\mathscr{A}_1^*)$. By Theorem 2, $T \in \mathrm{BKW}(\mathscr{A}; \{1, z\})$.

Next, let $T \in \text{BKW}(\mathscr{A}; \{1, z\})$. We may assume that $\|T\| = 1$. Let

$$\psi = T1 \in \mathscr{A}, \tag{1}$$

$$\phi_1 = Tz \in \mathscr{A}. \tag{2}$$

Now we shall prove

$$|\psi| = |\phi_1| = 1 \qquad \text{on } \Gamma. \tag{3}$$

Let $\zeta_0 \in \Gamma = \partial \mathscr{A}$. Then

$$\int_\Gamma dT^* \zeta_0 = (T^* \zeta_0)(1) = (T1)(\zeta_0) = \psi(\zeta_0), \tag{4}$$

$$\int_\Gamma z dT^* \zeta_0 = (Tz)(\zeta_0) = \phi_1(\zeta_0). \tag{5}$$

Since $S = \{1, z\}$ and $\|a + bz\|_\infty = |a| + |b|$ for $a, b \in$ C, we have

$$\sup \left\{ \left| \int_\Gamma s \, dT^* \zeta_0 \right| ; s \in \tilde{S}, \|s\|_\infty = 1 \right\} = \max \left\{ \left| \int_\Gamma dT^* \zeta_0 \right|, \left| \int_\Gamma z \, dT^* \zeta_0 \right| \right\}.$$

Hence, by Lemma 5, we have $\|T^* \zeta_0\| = 1$ and then

$$|\psi(\zeta_0)| = 1 \qquad \text{or} \qquad |\phi_1(\zeta_0)| = 1.$$

To prove (3), suppose not. Here we assume that $|\psi(\zeta_0)| = 1$ and $|\phi_1(\zeta_0)| < 1$. By the same way, we can work for the case $|\psi(\zeta_0)| < 1$ and $|\phi_1(\zeta_0)| = 1$. Since $|\overline{\psi(\zeta_0)} \, \phi_1(\zeta_0)| < 1$, it is not difficult to find many probability measures v on Γ of the form

$$v = a\,\delta_{t_1} + (1 - a)\,\delta_{t_2}, \qquad \text{where} \quad 0 < a < 1 \qquad \text{and} \qquad t_1, t_2 \in \Gamma,$$

such that

$$\int_\Gamma z \, dv = \overline{\psi(\zeta_0)} \, \phi_1(\zeta_0).$$

Let $\mu = \psi(\zeta_0)\, v$. Then $\|\mu\| = 1$, and by (4) and (5) we have

$$\int_\Gamma d\mu = \psi(\zeta_0) = \int_\Gamma dT^* \zeta_0 \qquad \text{and} \qquad \int_\Gamma z \, d\mu = \phi_1(\zeta_0) = \int_\Gamma z \, dT^* \zeta_0. \tag{6}$$

By the above construction of μ, it is easy to see the existence of μ_1 and μ_2 satisfying (6) and $\int_\Gamma z^2 \, d\mu_1 \neq \int_\Gamma z^2 \, d\mu_2$. Then by Lemma 5, $T \notin \text{BKW}(\mathscr{A}; \{1, z\})$. Hence we get (3).

Now by (3), (4), and (5), for every $\zeta \in \Gamma$ we obtain

$$T^*\zeta = \psi(\zeta)\, \delta_{\phi(\zeta)}, \qquad \text{where} \quad \phi(\zeta) = \phi_1(\zeta)/\psi(\zeta).$$

Therefore we have

$$(Tf)(\zeta) = (T^*\zeta)(f) = \psi(\zeta)(f \circ \phi)(\zeta) \qquad \text{for} \quad f \in \mathscr{A}.$$

When $f = z^n$, we have

$$\psi\, \phi^n = \phi_1^n / \psi^{n-1} \in \mathscr{A}, \qquad n \in \mathbf{N}. \tag{7}$$

By (1), (2), and (3), ψ and ϕ_1 are finite Blaschke products. Then by (7), we obtain $\phi_1/\psi \in \mathscr{A}$ and that $\phi = \phi_1/\psi$ is a finite Blaschke product. This completes the proof.

By the same argument, we can get the following.

THEOREM 4. BKW$(C(\Gamma); \{1, z\}) = \{aM_\psi C_\phi; \psi, \phi \in C(\Gamma), |\psi| = |\phi| = 1$ *on* Γ, $a \in C\}$.

Also in the same way as the proof of Theorem 3, we obtain the following.

THEOREM 5. BKW$(\mathscr{A}; \{1, z^n\}) = \{0\}$ *and* BKW$(C(\Gamma); \{1, z^n\}) = \{0\}$ *for* $n \geqslant 2$.

Proof. Let $n \geqslant 2$. We only prove the first assertion. Let $T \in$ BKW$(\mathscr{A}; \{1, z^n\})$ and $\|T\| = 1$. Let $\psi = T1 \in \mathscr{A}$ and $\phi = Tz^n \in \mathscr{A}$. In the same way as the proof of Theorem 3,

$$|\psi| = |\phi| = 1 \qquad \text{on} \quad \Gamma.$$

Let $\zeta \in \Gamma$. Then

$$\int_\Gamma dT^*\,\zeta = \psi(\zeta) \qquad \text{and} \qquad \int_\Gamma z^n\, dT^*\,\zeta = \phi(\zeta).$$

Let $\zeta_1, \quad ..., \zeta_n$ be the distinct points in Γ such that

$$\zeta_j^n = \overline{\psi(\zeta)}\, \phi(\zeta) \qquad \text{for} \quad 1 \leqslant j \leqslant n.$$

For $a = (a_1, ..., a_n)$ with $a_j \geqslant 0$ and $\sum_{j=1}^n a_j = 1$, let

$$\mu_a = \psi(\zeta) \left(\sum_{j=1}^n a_j\, \delta_{\zeta_j} \right).$$

Then $\|\mu_a\| = 1 = \|T^*\zeta\|$, and

$$\int_\Gamma d\mu_a = \psi(\zeta) \qquad \text{and} \qquad \int_\Gamma z^n \, d\mu_a = \phi(\zeta).$$

It is not difficult to see that $\int_\Gamma z^{n+1} \, d\mu_a \neq \int_\Gamma z^{n+1} \, dT^*\zeta$ for some $a = (a_1, ..., a_n)$. Hence, by Lemma 5, we have $T \notin \mathrm{BKW}(\mathscr{A}; \{1, z^n\})$. This is a contradiction.

Let $T \in \mathrm{BKW}(\mathscr{A}, A; \{1, z\})$ with $\|T\| = 1$ for a function algebra A. Then in the same way as the proof of Theorem 3, we have

$$|T1| = |Tz| = 1 \qquad \text{on } \partial A,$$
$$Tf = \psi(f \circ \phi) \qquad \text{for every} \quad f \in \mathscr{A},$$

where $\psi = T1 \in A$, $\phi = Tz/T1$, and

$$(T1)(Tz/T1)^n = (Tz)^n/(T1)^{n-1} \in A \qquad \text{for} \quad n \in \mathbf{N}.$$

Here the reader may expect that $\phi \in A$ and $T = M_\psi C_\phi$ for $\psi, \phi \in A$. But it is not so.

Example 2. Let $H^\infty(D)$ be the space of all bounded analytic functions on D. For each $f \in H^\infty(D)$, there exists a radial limit function $f(e^{i\theta})$ for almost every $e^{i\theta} \in \Gamma$. We denote by $H^\infty(\Gamma)$ the space of these radial limit functions, and we identify $H^\infty(\Gamma)$ with $H^\infty(D)$. Let $A = H^\infty(\Gamma) + C(\Gamma)$. Then A is an essential supremum norm closed subalgebra of $L^\infty(\Gamma)$, hence A is a function algebra (see [6]). By [8], there exist inner functions q_1 and q_2 such that $q_2/q_1 \notin A$ and $q_2^{n+1}/q_1^n \in A$ for every $n \in \mathbf{N}$. Let

$$Tf = q_1(f \circ (q_2/q_1)) \qquad \text{for} \quad f \in \mathscr{A}.$$

Then $T \in B(\mathscr{A}, A)$. By the first paragraph of the proof of Theorem 3, we have $T \in \mathrm{BKW}(\mathscr{A}, A; \{1, z\})$. By the definition of T, $T1 = q_1$ and $Tz = q_2$, so that $Tz/T1 = q_2/q_1 \notin A$. But $(Tz)^n/(T1)^{n-1} = q_2^n/q_1^{n-1} \in A$ for $n \in \mathbf{N}$.

Finally, we discuss $\mathrm{BKW}(\mathscr{A}; \{1, z, z^2\})$. In [16, Theorem 1], Takahasi proved that if ϕ_1 and ϕ_2 are finite Blaschke products, and a_1, a_2 are positive numbers, then $a_1 C_{\phi_1} + a_2 C_{\phi_2} \in \mathrm{BKW}(\mathscr{A}; \{1, z, z^2\})$. He actually proved that

Lemma 6. $\{a_1\delta_{z_1} + a_2\delta_{z_2}; z_1, z_2 \in \Gamma, a_1 + a_2 = 1, a_1, a_2 \geqslant 0\} \subset U_{\{1,z,z^2\}}(M_1(\Gamma))$.

By Takahasi's result and Theorem 3, we have a conjecture that if $T \in \mathrm{BKW}(\mathscr{A}; \{1, z, z^2\})$, then T has the form

$$T = a_1 M_{\psi_1} C_{\phi_1} + a_2 M_{\psi_2} C_{\phi_2},$$

where ψ_i and ϕ_i, $i=1,2$, are finite Blaschke products and $a_1, a_2 \in C$. We show two examples which say the above conjecture is not true.

EXAMPLE 3. Let $\lambda(e^{i\theta})=(e^{i\theta}+e^{-i\theta}+2)/4$, $e^{i\theta}\in \Gamma$. Then $0\leqslant \lambda \leqslant 1$ on Γ. Let

$$(Tf)(e^{i\theta})=\lambda(e^{i\theta})\, f(e^{i\theta})+(1-\lambda(e^{i\theta}))\, f(e^{2i\theta})=(\lambda C_z+(1-\lambda)\, C_{z^2})(f)(e^{i\theta})$$

for $f\in \mathscr{A}$. Then $|(Tf)(e^{i\theta})|\leqslant \|f\|_\infty$. For $f\in \mathscr{A}$, we can write as $f(e^{i\theta})=f(0)+e^{i\theta}h(e^{i\theta})$ for some $f\in \mathscr{A}$. Then

$$\begin{aligned}(Tf)(e^{i\theta}) &= f(e^{2i\theta})+\lambda(e^{i\theta})(f(e^{i\theta})-f(e^{2i\theta}))\\ &= f(e^{2i\theta})+\lambda(e^{i\theta})\, e^{i\theta}(h(e^{i\theta})-e^{i\theta}\, h(e^{2i\theta}))\\ &= f(e^{2i\theta})+(h(e^{i\theta})-e^{i\theta}\, h(e^{2i\theta}))(e^{2i\theta}+1+2e^{i\theta})/4\\ &\in \mathscr{A}.\end{aligned}$$

Hence $T\in B(\mathscr{A})$, $\|T\|=1$, and $T1=1$. By the definition of T,

$$T^*\, e^{i\theta}=(e^{i\theta}+e^{-i\theta}+2)/4\ \delta_{e^{i\theta}}+(1-(e^{i\theta}+e^{-i\theta}+2)/4)\ \delta_{e^{2i\theta}}.$$

Then by Lemma 6, $T^*\, e^{i\theta}\in U_{\{1,z,z^2\}}(M_1(\Gamma))$ for every $e^{i\theta}\in \Gamma$. Hence by Theorem 2, we have $T\in \mathrm{BKW}(\mathscr{A};\{1,z,z^2\})$. We note that $\lambda\notin \mathscr{A}$.

EXAMPLE 4. We consider that $\{e^{i\theta};\ -\pi<\theta\leqslant\pi\}=\Gamma$. Let $\lambda(e^{i\theta})=(e^{i\theta/2}+e^{-i\theta/2}+2)/4$, $e^{i\theta}\in\Gamma$. Then $0\leqslant\lambda\leqslant 1$ on Γ. Let

$$\begin{aligned}(Tf)(e^{i\theta}) &= \lambda(e^{i\theta})\, f(e^{i\theta/2})+(1-\lambda(e^{i\theta}))\, f(-e^{i\theta/2})\\ &= (\lambda\, C_{\sqrt{z}}+(1-\lambda)\, C_{-\sqrt{z}})(f)(e^{i\theta})\end{aligned}$$

for $f\in\mathscr{A}$. We note that the function $\sqrt{z}=e^{i\theta/2}$ on Γ is not continuous at $z=-1$. We note that $T1=1$. For $n\geqslant 1$, we have

$$\begin{aligned}(Tz^{2n})(e^{i\theta}) &= \lambda(e^{i\theta})(e^{in\theta}-e^{in\theta})+e^{in\theta}=e^{in\theta}=z^n\in\mathscr{A},\\ (Tz^{2n+1})(e^{i\theta}) &= 2\lambda(e^{i\theta})\, e^{in\theta}\, e^{i\theta/2}-e^{in\theta}\, e^{i\theta/2}\\ &= (e^{i(n+1)\theta}+e^{in\theta})/2\\ &= (z^{n+1}+z^n)/2\in\mathscr{A}.\end{aligned}$$

For each $f\in\mathscr{A}$, there exists a sequence of analytic polynomials $\{p_n\}_n$ such that $\|p_n-f\|_\infty\to 0$. By the above, $Tp_n\in\mathscr{A}$. By the definition of T,

$\|Tp_n - Tf\|_\infty \to 0$ as $n \to \infty$. Hence $Tf \in \mathscr{A}$ for every $f \in \mathscr{A}$. Then in the same way as Example 3, $T \in$ BKW$(\mathscr{A}; \{1, z, z^2\})$, $\|T\| = 1$, and $T1 = 1$.

By the above two examples, we think that it is fairly difficult to give a complete description of operators in BKW$(\mathscr{A}; \{1, z, z^2\})$.

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